Linear differential equations

One week

Contents

Module **2**

Motivation **2**

Learning outcomes **2**

1 Linear differential equation of order 1 **3**
Exercise 1.1 3
Exercise 1.2 3

2 Linear differential equation of order 2 with constant coefficients **4**
Exercise 2.3 4
Exercise 2.4 4

Module

This chapter is part of the APEF module: Analysis, Polynomials, Differential Equations, Functions. It is in the B3.

The **prerequisites** are:

- Module [MATH-S1-1-OBM] (seminar).
- Solving a 2nd degree equation.
- Finding the derivatives of usual functions and of their compositions.
- Finding primitive functions of usual functions (OBM)

Motivation

In a differential equation, the "unknown variable" is a function. Solving such an equation consists in finding the set of all the functions which satisfy a given relation between the function and its derivatives.

Some differential equations are really difficult to solve. For most of them, the solutions cannot be expressed with the usual functions and the only way to solve them is to run simulations in order to approximate their values. However, during this semester, we will focus on simple ones: the "order 1 linear equations" and the "order 2 linear equations with constant coefficients".

These equations are used in many fields of physics and electronics. Studying the movement of an object in the space leads to solve an order 2 differential equation: a relation between the position, the velocity and the acceleration of the object. Similarly, the oscillations of a pendulum, the study of an RLC circuit or modeling population dynamics, for example, involve such equations.

Learning outcomes

At the end of this chapter, you are expected to be able to solve an order 1 linear differential equation and an order 2 linear differential equation with constant coefficients.

These abilities involve:

- Solving an homogeneous linear equation of order 1, which will require finding primitive functions.
- Find an evident solution of an order 1 linear equation when there is one.
- Use the "variation of the parameter" method to find a particular solution of an order 1 equation.
- Solve an order 2 homogeneous linear differential equation with constant coefficients.
- Find an evident solution of an order 2 linear equation when there is one.
- Find a particular solution of an order 2 homogeneous linear differential equation with constant coefficients, when the right hand side is a polynomial, or a polynomial multiplied by an exponential function.

1 Linear differential equation of order 1

Exercise 1.1
Solve the following differential equations:

1. $xy' - 2y = 0$ on $\mathbb{R}_+^*$.
2. $(x^2+1)y' - y = 1$ on $\mathbb{R}$.
3. $x\ln(x)y' - y = 4$ on $]1, +\infty[$.
4. $y' + y = e^x - 1$ on $\mathbb{R}$.
5. ***(Homework)*** $y' - 2xy = (1-2x)e^x$ on $\mathbb{R}$.
6. $y' - \dfrac{2y}{x+1} = (x+1)^3$ on $]-1, +\infty[$.
7. $(1+x^2)y' + xy = 3x^3 + 3x$ on $\mathbb{R}$.

   For the particular solution, look for a polynomial function of degree 2.
8. $\cos(t)y' - \sin(t)y = 1$ on $\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[$ with the initial condition $y(0) = 2$.

Exercise 1.2
Lea takes a mortgage of an amount C_0 (in euros) at a bank. That is, the bank lends to Lea the capital C_0. At each time $t > 0$, let $C(t)$ denote the amount of capital remaining due by Lea. The function $t \longmapsto C(t)$ satisfies the following rules:

- The banks charges a constant interest r (per year) during the whole credit duration.
- Lea repays the mortgage: each year, she pays a constants value k (in euros).

The function $t \longmapsto C(t)$ hence satisfies the following differential equation:

$$\frac{\mathrm{d}C}{\mathrm{d}t} = rC - k$$

where:

- $C(t)$ is the amount of capital remaining due by Lea at time t
- r is the interest rate per year
- k is the year repayment
- $C(0) = C_0$ is the initial capital that has been lent to Lea.

1. Justify the differential equation satisfied by the function C.
2. Solve this differential equation.
3. Explain why we can impose that $k > rC_0$.
4. Find the duration T required for Lea to repay the whole mortgage.
5. Numerical case: assume that the amount of the mortgage is 200000 euros, the interest rate is 3% per year and that Lea repays 500 euros each month. How much time will Lea need to repay the whole mortgage?

2 Linear differential equation of order 2 with constant coefficients

Exercise 2.3

Solve on $\mathbb{R}$ the following differential equations:

1. $y'' - 2y' + 10y = -5$
2. $y'' - y' - 2y = -x^2 - 3x$
3. $y'' + 2y' + 5y = xe^x$
4. ***(Homework)*** $y'' - 5y' + 6y = e^{2x}$
5. $y'' - 4y' + 4y = xe^{2x}$
6. ***(Homework)*** $y'' + y = e^x$

Exercise 2.4

(Homework) Solve on $\mathbb{R}$ the following differential equations:

1. $2y'' + 6y' + 5y = 10x^3 + 36x^2 + 29x + 1$
2. $y'' + 2y' - 3y = (12x^2 + 16x + 14)e^{3x}$
3. $4y'' + 16y' + 16y = 8e^{-2x}$
4. $y'' - y = 2e^x$